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Birkhoff Theorem in Self-Creation Cosmology

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An analogue of Birkhoff's theorem of general relativity holds for electromagnetic fields in the self-creation cosmology proposed by Barber when the scalar field is independent of time.

1. INTRODUCTION

Many theories have been proposed as alternatives to Einstein's theory of general relativity, the most important among them being the scalar-tensor theory proposed by Brans and Dicke (1961). This theory develops Mach's principle in a relativistic framework by assuming that inertial masses of fundamental particles are not constant, but are dependent upon the interaction of the particles with some cosmic scalar field coupled to the large-scale distribution of matter in motion. The Brans-Dicke theory does not allow the scalar field to otherwise interact with fundamental particles and photons. By allowing the scalar field to interact with particle and photon momentum 4-vectors and thus modifying the Brans-Dicke theory to allow for the continuous creation of matter, Barber (1982), has developed a continuous-creation theory. In this theory the universe is seen to be created out of self-contained gravitational, scalar and matter fields. The theory predicts local effects that are within the limits already observed. In the limit $\alpha \to 0$ the theory approaches standard general relativity theory in every respect. Recently, we have discussed some general results on spatially homogeneous stationary cosmological models in this theory (Singh and Singh, 1985).

Birkhoff's theorem states that for $r > 2m$ every spherically symmetric solution of the Einstein vacuum field equations is static (Birkhoff, 1927; Straumann, 1984). This result holds in the presence of electromagnetic fields also (Hoffmann, 1932; Das, 1960). This theorem also has been found to be

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valid in other theories of gravitation (Reddy, 1977; Singh and Rai, 1979; Dhuttachaudhury and Bhattacharya, 1980) under the constraint $\dot{\phi}=0$ (i.e., the scalar field is independent of time).

In this paper we show that a Birkhoff-type theorem of general relativity for electromagnetic fields is true in the self-creation theory proposed by Barber (1982) when the scalar field is independent of time. Although the conditions are stronger in the context of a theorem like Birkhoff's, given the paucity of investigations and solutions in this new theory, the authors have made a preliminary attempt in this direction. Further, it is a formidable task to solve equations (9)-(13). The effort is engaging our attention.

2. SPHERICALLY SYMMETRIC FIELDS IN SELF-CREATION THEORY

We start with the case where the scalar field is coupled to an electromagnetic field for which the Barber-Maxwell field equations are

$$R^i_j-\tfrac{1}{2}\delta^i_j R=-\frac{8\pi}{\phi}E^i_j-\frac{2}{3\alpha\phi}(\phi^{;i}_{;j}-\delta^i_j\Box\phi) \tag{1}$$

$$\Box\phi=0 \tag{2}$$

$$F^{ij}_{;j}=0 \tag{3}$$

$$F_{[ij,k]}=0 \tag{4}$$

where α is a coupling constant chosen so that the gravitational constant G, which is now a function of ϕ, can be defined as $G=1/\phi$. Comma and semicolon denote partial and covariant derivatives, respectively, and E^i_j is defined as

$$E^i_j=F^{ik}F_{jk}-\tfrac{1}{4}\delta^i_j F^{kl}F_{kl}$$

We consider the spherically symmetric metric in the form

$$ds^2=e^\nu\,dt^2-e^\lambda\,dr^2-r^2(d\theta^2+\sin^2\theta\,d\Phi^2) \tag{5}$$

where $\lambda=\lambda(r,t)$ and $\nu=\nu(r,t)$. Due to spherical symmetry

$$F_{12}=F_{13}=F_{24}=F_{34}=0; \qquad F_{14}\text{ and }F_{23}\neq 0 \tag{6}$$

Also spherical symmetry implies that $\phi=\phi(r,t)$.

In view of the metric (5) it follows from equations (3) and (4) that

$$F_{14}=(q/r^2)\,e^{(\lambda+\nu)/2} \tag{7}$$

$$F_{23}=m\sin\theta \tag{8}$$

where q and m are arbitrary constants and can be interpreted, respectively, as the charge and the magnetic pole strength of the point source. Now, for the metric (5), the Barber-Maxwell field equations (1) and (2) are

$$-e^{-\lambda}\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)+\frac{1}{r^2} = -\frac{8\pi(q^2+m^2)}{\phi r^4}-\frac{2}{3\alpha\phi}\left[2\,e^{-\lambda}\phi'\frac{1}{r}+e^{-\lambda}\phi'\frac{\nu'}{2}-e^{-\nu}\left(\ddot{\phi}-\frac{\dot{\nu}\dot{\phi}}{2}\right)\right] \tag{9}$$

$$-e^{-\lambda}\left(\frac{\nu''}{2}-\frac{\lambda'\nu'}{4}+\frac{\nu'^2}{4}+\frac{\nu'-\lambda'}{2}\right)+e^{-\nu}\left(\frac{\ddot{\lambda}}{2}+\frac{\dot{\lambda}^2}{4}-\frac{\dot{\lambda}\dot{\nu}}{4}\right) = -\frac{8\pi(q^2+m^2)}{\phi r^4}-\frac{2}{3\alpha\phi}\left[-e^{-\lambda}\left(\frac{\lambda'\phi'}{2}-\phi''\right)\right.$$
$$\left.-\frac{e^{-\nu}}{2}\dot{\phi}\dot{\lambda}+e^{-\lambda}\phi'\frac{1}{r}-e^{-\nu}\left(\ddot{\phi}-\frac{\dot{\nu}\dot{\phi}}{2}\right)+\frac{e^{-\lambda}}{2}-\phi'\nu'\right] \tag{10}$$

$$e^{-\lambda}\left(\frac{\lambda'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}=-\frac{8\pi(q^2+m^2)}{\phi r^4}-\frac{2}{3\alpha\phi}\left[-e^{-\lambda}\left(\frac{\phi'\lambda'}{2}-\phi''\right)-\frac{e^{-\nu}}{2}\dot{\phi}\dot{\lambda}+2\,e^{-\lambda}\phi'\frac{1}{r}\right] \tag{11}$$

$$-e^{-\lambda}\frac{\dot{\lambda}}{r}=-\frac{2\,e^{-\lambda}}{3\alpha\phi}\left(\frac{\dot{\phi}\nu'}{2}-\dot{\phi}'+\frac{\dot{\lambda}\phi'}{2}\right) \tag{12}$$

$$e^{-\lambda}\left(\frac{\lambda'\phi'}{2}-\phi''-\frac{\phi'\nu'}{2}-\frac{2\phi'}{r}\right)+e^{-\nu}\left(\ddot{\phi}-\frac{\dot{\nu}\dot{\phi}}{2}+\frac{\dot{\phi}\dot{\lambda}}{2}\right)=0 \tag{13}$$

where primes and dots denote partial derivatives with respect to r and t, respectively.

When the scalar field ϕ is a function of r alone,

$$\dot{\phi}=0 \tag{14}$$

Then from equation (12) we have

$$\dot{\lambda}\left(\frac{1}{r}-\frac{\phi'}{3\alpha\phi}\right)=0 \tag{15}$$

which implies that either

$$\dot{\lambda}=0 \tag{16}$$

or

$$\frac{1}{r}=\frac{\phi'}{3\alpha\phi}, \qquad \text{i.e.,} \qquad \phi=\phi_0 r^{3\alpha}, \quad \phi_0=\text{const} \tag{17}$$

Plugging the value of ϕ into the equations, one finds certain restrictions on the value of α.

When $\dot{\phi}=0$, from equations (9), (11), and (13) it follows that

$$1-\frac{2r\phi'}{3\alpha\phi}-\frac{r}{2}(\lambda'-\nu')=e^{\lambda}\left[1+\frac{8\pi(q^2+m^2)}{\phi r^2}\right] \tag{18}$$

Using (17) in equation (13), we get

$$\lambda'-\nu'=\frac{2}{r}(3\alpha+1) \tag{19}$$

Using (17) and (19) in equation (18), we have

$$e^{\lambda}\left[1+\frac{8\pi(q^2+m^2)}{\phi_0 r^{3\alpha+2}}\right]+3\alpha+2=0 \tag{20}$$

Thus λ is a function of r only. Therefore

$$\dot{\lambda}=0$$

Now differentiation of (19) with respect to t along with the use of (14) and (16) gives

$$\dot{\nu}'=0 \tag{21}$$

From this one has

$$\nu=f(r)+g(t) \tag{22}$$

where f and g are arbitrary functions of r and t, respectively. We can now use the transformation (Das, 1960)

$$dt'=e^{g(t)/2}\,dt$$

This turns ν into a function of r only. This, together (16), reduces the metric (5) to the static case. Hence, Birkhoff's theorem is valid in this theory under the restriction that the scalar field is independent of time. Further, following the method suggested in Straumann (1984, pp. 198 and 208–213) it can be shown that Birkhoff's theorem remains true even for $r\leq 2m$ when we assume that the scalar field is independent of time. However, when ϕ is a function of both r and t, Birkhoff's theorem is not valid in this theory.

Now we consider the line element in the form

$$ds^2=e^{\nu}\,dt^2-e^{\lambda}\,dr^2-y^2(d\theta^2+\sin^2\theta\,d\Phi^2), \qquad y=\text{const} \tag{23}$$

Proceeding along lines similar to that for the metric (5), when we assume $\dot{\phi}=0$ and use in equation corresponding to (12) (i.e., for G_{14}) we get $\phi=\text{const}$. This reduces the field equations to the Einstein–Maxwell theory, for which the Birkhoff theorem is well known.

3. CONCLUSIONS

We have shown that a Birkhoff-type theorem of general relativity is true in the presence of electromagnetic fields in the self-creation theory proposed by Barber (1982) when the scalar field is independent of time. This may possibly be due to the fact that the interaction of the time-dependent scalar field with the electromagnetic field stimulates electromagnetic monopole radiation.

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